Quantum Deformations of Space–Time SUSY and Noncommutative Superfield Theory

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Abstract

We review shortly present status of quantum deformations of Poincaré and conformal supersymmetries. After recalling the κ–deformation of D=4 Poincaré supersymmetries we describe the corresponding star product multiplication for chiral superfields. In order to describe the deformation of chiral vertices in momentum space the integration formula over κ–deformed chiral superspace is proposed.

1 Introduction

The noncommutative space–time coordinates were introduced as describing algebraically the quantum gravity corrections to commutative flat (Minkowski) background (see e.g. [1, 2]) as well as the modification of D–brane coordinates in the presence of external background tensor fields (e.g. $B_{\mu\nu}$ in $D = 10$ string theory; see [3]–[5]). We know well that both gravity and string theory have better properties (e.g. less divergent quantum perturbative expansions) after their supersymmetrization. It appears therefore reasonable, if not compelling, to consider the supersymmetric extensions of the noncommutative framework.

The generic relation for the noncommutative space–time generators $\widehat{x}_\mu$

$$[\widehat{x}_\mu, \widehat{x}_\nu] = i\Theta_{\mu\nu}(\widehat{x}) = i\left(\Theta_{\mu\nu} + \Theta^{\rho}_{\mu\nu}\widehat{x}_\rho + \ldots\right) \tag{1.1}$$

has been usually considered for constant value of the commutator (1.1), i.e. for $\Theta_{\mu\nu}(\widehat{x}) = \Theta_{\mu\nu}$. In such a case the multiplication of the fields $\phi_k(\widehat{x})$ depending on the noncommutative (Minkowski) space–time coordinates can be represented by noncommutative Moyal $*$–product of classical fields $\phi_k(x)$ on standard Minkowski space

$$\phi_k(\widehat{x})\phi_l(\widehat{x}) \longleftrightarrow \phi_k(x) * \phi_l(x) = \phi_k(y) e^{\frac{i}{2}\Theta^{\mu\nu}\frac{\partial}{\partial y_\mu}\frac{\partial}{y_\nu}} \phi_l(z)|_{x=y} \tag{1.2}$$

It appears that the relation (1.1) with constant $\Theta_{\mu\nu}$ can be consistently supersymmetrized (see e.g. [6]–[9]) by supplementing the standard relations for the odd Grassmann superspace coordinates (further we choose $D=4 \quad N=1$ SUSY and $\alpha, \beta = 1,2$).

$$\{\theta_\alpha, \theta_\beta\} = \{\theta_\alpha, \overline{\theta}_{\dot{\beta}}\} = \{\overline{\theta}_{\dot{\alpha}}, \overline{\theta}_{\dot{\beta}}\} = 0 \qquad [\widehat{x}_\mu, \theta_\alpha] = [\widehat{x}_\mu, \overline{\theta}_{\dot{\alpha}}] = 0 \tag{1.3}$$

Such a choice of superspace coordinates $(\widehat{x}_\mu, \theta_\alpha, \theta_{\dot{\alpha}})$ implies that the supersymmetry transformations remain classical:

$$\widehat{x}'_\mu = \widehat{x}_\mu - i\left(\overline{\epsilon}\sigma_k\theta_\alpha - \overline{\theta}\sigma_k\epsilon\right)$$

$$\theta'^{\alpha} = \theta_\alpha + \epsilon_\alpha \qquad \overline{\theta}'^{\dot{\alpha}} = \overline{\theta_{\dot{\alpha}}} + \overline{\epsilon_{\dot{\alpha}}} \tag{1.4}$$

i.e. the covariance requirements of deformed superspace formalism do not require the deformation of classical Poincaré supersymmetries[1].

Our aim here is to consider the case when the standard Poincaré supersymmetries can not be preserved. For this purpose we shall consider the case with linear Lie–algebraic commutator (1.1). Its supersymmetrization leads to the deformed superspace coordinates $\widehat{z}_A = (\widehat{x}_\mu, \widehat{\theta}_\alpha, \widehat{\overline{\theta}}_{\dot{\beta}})$ satisfying Lie superalgebra relation:

$$[\widehat{z}_A, \widehat{z}_B] = i\Theta_{AB}^{C}\,\widehat{z}_C \tag{1.5}$$

where Θ_{AB}^{C} satisfies graded Jacobi identity:

$$\Theta_{AB}^{\ D}\,\Theta_{CD}^{\ E} + \text{graded cycl. } (A,B,C) \qquad = 0 \tag{1.6}$$

It appears that in such a case for some choices of the "structure constants" $\Theta_{AB}^{\ C}$ one can find the deformed quantum $D=4$ Poincaré supergroup, which provide the relations (1.5) as describing the deformed translations and deformed supertranslations.

The plan of the paper is following: In Sect. 2 we shall briefly review the considered in literature quantum deformations of Poincaré and conformal supersymmetries. The list of these deformations written in explicite form as Hopf

[1] It should be stressted, however, that the introduction of constant tensor $\Theta_{\mu\nu}$ in (1.1) leads to breaking $(O(3,1) \to O(2) \times O(1,1)))$ of $D=4$ Lorentz symmetry. The way out is to consider $\Theta_{\mu\nu}$ as a constant field, with generator of Lorentz subalgebra containing contribution which rotates the $\Theta_{\mu\nu}$ components (see e.g. [10]). The relation (1.1) can be made covariant only for $D=2$ ($\Theta_{\mu\nu} \equiv \epsilon_{\mu\nu}$ for $D=2$); for 2+1 Euclidean case see [11]

algebras is quite short, and only the knowledge of large class of classical r–matrices shows that many quantum deformations should be still discovered. As the only nontrivial quantum deformation of $D = 4$ supersymmetry given in the literature is the so–called κ–deformation, obtained in 1993 [12]–[14].

In Sect. 3 we consider the Fourier supertransform of superfields in classical (undeformed) and κ–deformed form. We present also the integration formula over κ–deformed superspace, which provides the description in supermomentum space leading to the κ–deformed Feynmann superdiagrams.

In Sect. 4 we consider the κ–deformed superfield theory in chiral superspace. We introduce the $*$–product multiplication of κ–deformed superfields. It appears that there are two distinguished $*$–products, which both can be written in closed form: one described by standard supersymmetric extenion of CBA formula and other physical, providing the addition of fourmomenta and Grassmann momenta in terms of the coproduct formulae. In such a way we obtain the supersymmetric extension of two $*$–products, considered recently in [15].

In Sect. 5 we shall present some remarks and general diagram describing the deformation scheme of superfield theory.

2 Quantum Deformations of Space–Time Supersymmetries

There are two basic space–time symmetries in D dimensions:

- Conformal symmetries $O(D, 2)$, having another interpretation as anti–de–Sitter symmetries in $D + 1$ dimensions

- Poincaré symmetries $T^{D-1,1} \niplus O(D - 1, 1)$.

i) Quantum deformations of conformal supersymmetries.

The conformal symmetries can be supersymmetrized without introducing tensorial central charges in $D = 1, 2, 3, 4$ and 6. One gets:

$$D = 1 \quad : \quad O(2, 1) \longrightarrow OSp(N; 2|R) \quad \text{or} \quad SU(1, 1 : N)$$

$$D = 2 \quad : \quad O(2, 2) = O(1, 2) \otimes O(1, 2) \longrightarrow OSp(M; 2|R) \otimes OSp(N; 2|R)$$

$$D = 3 \qquad O(3, 2) \longrightarrow OSp(N; 4|R)$$

$$D = 4 \qquad O(4, 2) \longrightarrow SU(2, 2; N)$$

$$D = 6 \qquad O(6, 2) \longrightarrow U_\alpha U(4; N|H)$$

All conformal supersymmetries listed above are described by simple Lie superalgebras. It is well–known that for every simple Lie superalgebra one can introduce the q–deformed Cartan–Chevaley basis describing quantum (Hopf–algebraic) Drinfeld–Jimbo deformation [16, 17]. These q–deformed relations

have been explicitly written in physical basis of conformal superalgebra in different dimensions (see e.g. [18]–[21]). It is easy to see that the deformation parameter q appears as dimensionless.

It follows, however, that there is another class of deformations of conformal and superconformal symmetries, with dimensionfull parameter κ, playing the role of geometric fundamental mass. For $D = 1$ one can show that the Jordanian deformation of $SL(2;R) \simeq O(2,1)$ describes the κ–deformation of $D = 1$ conformal algebra [22]. This result can be extended supersymmetrically, with the following classical $\widehat{r}$–matrix describing Jordanian deformation $U_\kappa(OSp(1;2|R)$ [23]

$$
\begin{array}{ccc}
r = \dfrac{1}{\kappa}\, h \wedge e & \overset{SUSY}{\Longrightarrow} & r = \dfrac{1}{\kappa}\left(h \wedge e + Q^+ \wedge Q^+\right) \\
\text{Jordanian deformation} & & \text{Jordanian deformation} \\
\text{of } Sp(2;R) \simeq O(2;1;R) & & \text{of } OSp(1,2;R) \\
(D = 1 \quad \text{conformal}) & & (D = 1 \quad \text{superconformal})
\end{array}
\qquad (2.1)
$$

The $OSp(1;2;R)$ Jordanian classical $\widehat{r}$–matrix can be quantized by the twist method. Semi–closed form for the twist function has been obtained in [24].

It appears that one can extend the Jordanian deformations of $D = 1$ conformal algebra to $D > 1$; for $D = 3$ and $D = 4$ the extended Jordanian classical r–matrices were given in [22]. It should be also mentioned that the generalized Jordanian deformation of $D = 3$ conformal $O(3,2)$ algebra has been obtained in full Hopf–algebraic form [25]. The extension of Jordanian deformation of $OSp(1,2;R)$ for $D > 1$ superconformal algebras is not known even in its infinitesimal form given by classical r–matrices.

ii) Quantum deformations of Poincaré supersymmetries.

Contrary to DJ scheme for simple Lie (super)algebras it does not exist a systematic way of obtaining quantum deformations of non–semisimple Lie (super)algebras. A natural framework for the description of deformed semi–direct products, like quantum Poincaré algebra, are the noncocommutative bicrossproduct Hopf algebras (see e.g. [26]). It appears however that in the literature it has not been formulated any effective scheme describing these quantum bicrossproducts.

One explicite example of quantum deformation of $D = 4$ Poincaré superalgebra and its dual $D = 4$ Poincaré group in form of graded bicrossproduct Hopf algebra was given in [14]. By means of quantum contraction of q–deformed $N = 1$ anti–de–Sitter superalgebra $U_q\,(OSp(1|4))$ there was obtained in [12] the κ–deformed $D = 4$ Poincaré subalgebra $U_\kappa(\mathcal{P}_{4;1})$. Subsequently by nonlinear change of generators the quantum superalgebra $U_\kappa(\mathcal{P}_{4;1})$ was written in chiral bicrossproduct basis [13]. The κ–deformed Poincaré subalgebra is given by the deformation of the following graded cross–product [2]

$$
p_{4;1} = \Big(SL(2;C) \oplus (\overline{SL(2;C)} \supset\!\!\!\!\!\! + \, \overline{T}_{0;2} \Big) \ltimes T_{4;2} \qquad (2.2)
$$

[2]In [13] for the crossproduct formula describing $D = 4$ superPoincaré algebra the following notation was used: $p_{4;1} = O(1,3;2) \ltimes T_{4,2}$. In the notation (2.2) proposed in present paper the extension of Lorentz algebra by odd generators is described more accurately.

where the generators of $SL(2;C)$ are given by two–spinor generators $M_{\alpha\beta} = \frac{1}{8}(\sigma^{\mu\nu})_{\alpha\beta}M_{\mu\nu}$, the generators of $(\overline{SL(2;C)}$ by $M_{\dot\alpha\dot\beta} = M^*_{\alpha\beta} = \frac{1}{8}\sigma^{\mu\nu}_{\dot\alpha\dot\beta}M_{\mu\nu}$, $\overline{T}_{0;2}$ describes two antichiral supercharges $\overline{Q}_{\dot\alpha}$, and $T_{4;2}$ the graded Abelian superalgebra

$$T_{4;2}: \qquad [P_\mu, P_\nu] = [P_\mu, Q_\alpha] = \{Q_\alpha, Q_\beta\} = 0 \tag{2.3}$$

The relations (2.3) describe the algebra of generators of translations and supertranslations in chiral superspace. The algebra $(SL(2;c) \oplus (\overline{SL(2;c)} \ni\!\!\!\!\supset \overline{T}_{0;2})$ has the form

$$\begin{aligned} sl(2;c): \qquad [M_{\alpha\beta}, M_{\gamma\delta}] &= \epsilon_{\alpha\gamma}M_{\beta\delta} - \epsilon_{\beta\gamma}M_{\alpha\delta} \qquad \text{(2.4a)}\\ &\quad + c_{\beta\delta}M_{\alpha\gamma} - \epsilon_{\alpha\delta}M_{\beta\gamma} \\ \overline{sl(2;c)} \ni\!\!\!\!\supset \overline{T}_{0;2}: \qquad [M_{\dot\alpha\dot\beta}, M_{\dot\gamma\dot\delta}] &= \epsilon_{\dot\alpha\dot\gamma}M_{\dot\beta\dot\delta} - \epsilon_{\dot\beta\dot\gamma}M_{\dot\alpha\dot\delta} \\ &\quad + \epsilon_{\dot\beta\dot\delta}M_{\dot\alpha\dot\gamma} - \epsilon_{\dot\alpha\dot\delta}M_{\dot\beta\dot\gamma} \\ [M_{\dot\alpha\dot\beta}, Q_{\dot\gamma}] &= \epsilon_{\dot\alpha\dot\gamma}Q_{\dot\beta} - \epsilon_{\dot\beta\dot\gamma}Q_{\dot\alpha} \\ \{Q_{\dot\alpha}, Q_{\dot\beta}\} &= 0 \qquad \text{(2.4b)} \end{aligned}$$

It should be observed that in the cross-product (2.2) the basic supersymmetry algebra $\{Q_\alpha, Q_{\dot\beta}\} = 2(\sigma^\mu p_\mu)_{\alpha\dot\beta}$ is the one belonging to the cross–relations.

The κ–deformed bicrossproduct is given by the formula

$$U_\kappa(p_{4;2}) = (SL(2;c) \oplus \overline{SL(2;c)} \ni\!\!\!\!\supset \overline{T}_{0;2}) \bowtie T^\kappa_{4;2} \tag{2.5}$$

The relations (2.3) and (2.4a)–(2.4b) remain valid but $T^\kappa_{4;2}$ describes now the Hopf algebra with deformed coproducts:

$$\begin{aligned} \Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 \\ \Delta P_i &= P_i \otimes e^{-\frac{P_0}{\kappa}} + 1 \otimes P_i \\ \Delta Q_\alpha &= Q_\alpha \otimes e^{-\frac{P_0}{2\kappa}} + 1 \otimes Q_\alpha \end{aligned} \tag{2.6}$$

The cross–relations are the following ($M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, N_i = M_{i0}$):

$$\begin{aligned} [M_i, P_j] &= i\epsilon_{ijk}P_k \qquad [M_i, P_0] = 0 \\ [N_i, P_j] &= i\delta_{ij}[\frac{\kappa}{2}(1 - e^{-\frac{2P_0}{\kappa}} + \frac{1}{2\kappa}\vec{P}^{\,2}) + \frac{1}{\kappa}P_iP_j] \\ [N_i, P_0] &= iP_i \end{aligned} \tag{2.7}$$

and

$$[M_i, Q_\alpha] = -\frac{1}{2}(\sigma_i)_\alpha{}^\beta Q_\beta$$

$$
\begin{aligned}
[N_i, Q_\alpha] &= \frac{1}{2} i\, e^{-\frac{P_0}{\kappa}} (\sigma_i)_\alpha{}^\beta Q_\beta + \frac{1}{2\kappa}\, \epsilon_{ijk} P_j (\sigma_k)_\alpha{}^\beta Q_\beta \\
\{Q_\alpha, Q_{\dot\beta}\} &= 4\kappa \delta_{\alpha\dot\beta} \sinh \frac{P_0}{2\kappa} - 2 e^{\frac{P_0}{2\kappa}} p_i (\sigma_i)_{\alpha\dot\beta}
\end{aligned} \tag{2.8}
$$

The notion of bicrossproduct (2.5) implies also the modification of primitive coproducts for $SL(2;c) \oplus \overline{SL(2;c)} \ni\!\!\!\!\supset \overline{T}_{0;2}$ generators. One gets:

$$
\begin{aligned}
\Delta M_i &= M_i \otimes 1 + 1 \otimes M_i \\
\Delta N_i &= N_l \otimes 1 + e^{-\frac{P_0}{\kappa}} \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk}\, P_j \otimes M_k \\
&\quad - \frac{i}{4\kappa} (\sigma_i)_{\alpha\dot\beta}\, Q_\alpha \otimes e^{\frac{P_0}{\kappa}} Q_{\dot\beta} \\
\Delta Q_j &= Q_{\dot\alpha} \otimes 1 + e^{\frac{P_0}{2\kappa}} \otimes Q_{\dot\alpha}
\end{aligned} \tag{2.9}
$$

It appears that the classical $N = 1$ $D = 4$ Poincaré superalgebra can be put as well in the form

$$
p_{4;1} = (SL(2;c) \ni\!\!\!\!\supset T_{0;2}) \oplus (\overline{SL(2;c)} \ltimes \overline{T}_{4;2} \tag{2.10}
$$

where $\overline{T}{}^{\,0}_{4;2}$ describe the translation and supertranslation generators $(P_\kappa, Q_{\dot\alpha})$. Subsequently the κ–deformation of $D = 4$ $N = 1$ Poincaré superalgebra can be obtained by deforming (2.10) into graded bicrossproduct Hopf superalgebra

$$
U_\kappa(p_{4;1}) = (SL(2;c) \ni\!\!\!\!\supset T_{0;2}) \oplus (\overline{SL(2;c)} \bowtie \overline{T}{}^{\,\kappa}_{4;2} \tag{2.11}
$$

In order to describe the κ–deformed chiral superspace one should consider the Hopf superalgebra $\widetilde{T}{}^{\kappa}_{4;2}$ obtained by dualization of the relations (2.3) and (2.6), and describing by functions $C(\widehat{z}_A)$ on κ–deformed chiral superspace $\widehat{z}_A = (\widehat{z}_\mu, \widehat{\theta}_\alpha)$, where $\widehat{z}_\mu$ denotes the complex space–time coordinates. One obtains the following set of relations:

$$
\begin{aligned}
&[\widehat{z}_0, \widehat{z}_i] = \frac{i}{\kappa} \widehat{z}_i \qquad [\widehat{z}_i, \widehat{z}_j] = 0 \\
&[\widehat{z}_0, \widehat{\theta}_\alpha] = \frac{i}{2\kappa} \widehat{\theta}_\alpha \qquad [\widehat{z}_i, \widehat{\theta}_\alpha] = 0 \\
&\{\widehat{\theta}_\alpha, \widehat{\theta}_\beta\} = 0
\end{aligned} \tag{2.12a}
$$

and the primitive coproducts:

$$
\Delta \widehat{z}_\mu = \widehat{z}_\mu \otimes 1 + 1 \otimes \widehat{z}_\mu \qquad \Delta \widehat{\theta}_\alpha = \widehat{\theta}_\alpha \otimes 1 + 1 \otimes \widehat{\theta}_\alpha \tag{2.12b}
$$

The κ–deformed chiral superfield theory is obtained by cosidering suitably ordered superfields. In the following Section we shall consider the superFourier transform of deformed superfields and consider the κ–deformed chiral superfield theory.

3 Fourier Supertransforms and κ–deformed Berezin Integration

i) Fourier supertransform on classical superspace.

The superfields are defined as functions on superspace. Here we shall restrict ourselves to $D = 4$ chiral superspace $z_A = (z_\mu, \theta_\alpha)$ $(\mu = 0, 1, 2, 3; \alpha = 1, 2)$ and to chiral superfields $\Phi(z, \theta)$.

The Fourier supertransform of the chiral superfield and its inverse take the form:

$$\Phi(x, \theta) = \frac{1}{(2\pi)^2} \int d^4p\, d^2\eta\, \widetilde{\Phi}(p, \eta) e^{i(px+\eta\theta)} \tag{3.1a}$$

$$\widetilde{\Phi}(p, \eta) = \frac{1}{(2\pi)^2} \int d^4x\, d^2\theta\, \Phi(x, \theta) e^{-i(px+\eta\theta)} \tag{3.1b}$$

The Fourier supertransforms were considered firstly in [29, 30]. It appears that the set of even and odd variables $(z_\mu, \theta_\alpha; p_\mu, \eta_\alpha)$ describes the superphase space, with Grassmann variables η_α describing "odd momenta". The Berezin integration rules are valid in both odd position and momentum sectors:

$$\int d^2\theta = \int d^2\theta\, \theta_\alpha = 0 \qquad \frac{1}{2} \int d^2\theta\, \theta_\alpha\, \theta^\alpha = 1 \tag{3.2a}$$

$$\int d^2\eta = \int d^2\eta\, \eta_\alpha = 0 \qquad \frac{1}{2} \int d^2\theta\, \eta_\alpha\, \eta^\alpha = 1 \tag{3.2b}$$

where $\eta^\alpha = \epsilon^{\alpha\beta}\eta_\beta$ and $\eta_\alpha\eta^\alpha = 2\eta_1\eta_2$. It is easy to see that $\theta^2 = \frac{1}{2}\theta_\alpha\theta^\alpha$ $\eta^2 = \frac{1}{2}\eta_\alpha\eta^\alpha$ play the role of Dirac deltas, because

$$\int d^2\theta\, \theta^2\, \Phi(z, \theta) = \Phi(z, \theta) \mid_{\theta=0} \tag{3.3a}$$

$$\int d^2\eta\, \eta^2\, \widetilde{\Phi}(p, \eta) = \widetilde{\Phi}(p, \eta) \mid_{\eta=0} \tag{3.3b}$$

The formulae (3.1a)–(3.1b) in component formalism

$$\Phi(z, \theta) = \Phi(z) + \Psi^\alpha(z)\theta_\alpha + F(z)\theta^2 \tag{3.4a}$$

lead to

$$\widetilde{\Phi}(p, \eta) = \widetilde{F}(p) - \widetilde{\Psi}^\nu(p)\eta_\nu - \widetilde{\Phi}(p)\eta^2 \tag{3.4b}$$

Let us consider for example the chiral vertex $\Phi^3(z,\theta)$, present in Wess–Zumino model. This vertex can be written in momentum superspace as follows:

$$\int d^4z\, d^2\theta\, \Phi^3(z,\theta) = \int d^4p_1 \dots d^4p_3\, d^2\eta_1 \dots d^2\eta_3$$
$$\cdot\Phi(p_1,\eta_1)\,\Phi(p_2,\eta_2)\,\Phi(p_3,\eta_3)\delta^4(p_1+p_2+p_3)(\eta_1+\eta_2+\eta_3)^2 \quad (3.5)$$

We see therefore that in Feynmann superdiagrams the chiral vertex (3.5) will be represented by the product of Dirac deltas describing the conservation at the vertex of the fourmomenta as well as the Grassmann odd momenta.

ii) Fourier supertransform on κ–deformed superspace.

Following the formulae (2.12a)–(2.12b) we obtain the supersymmetric extension of of κ–deformed Minkowski space to κ–deformed superspace $\widehat{x}_\mu \longrightarrow (\widehat{x}_\mu, \widehat{\theta}_\alpha)$. The ordered superexponential is defined as follows:

$$:e^{i(p_\mu \widehat{z}^\mu + \eta_\alpha \widehat{\theta}^\alpha)} := e^{-ip_0\widehat{z}_0}\, e^{i(\vec{p}\vec{z} + \eta^\alpha \widehat{\theta}_\alpha)} \quad (3.6)$$

where (p_μ, θ_α) satisfy the Abelian graded algebra)2.3), i.e.

$$[p_\mu, p_\nu] = [p_\mu, \eta_\alpha] = \{\eta_\alpha, \eta_\beta\} = 0 \quad (3.7)$$

From the formulae (2.2)–(2.3) and (3.6)–(3.7) follows that:

$$:e^{i(p_\mu \widehat{z}^\mu + \eta_\alpha \widehat{\theta}^\alpha)} :: e^{i(p'_\mu \widehat{z}^\mu + \eta'_\alpha \widehat{\theta}^\alpha)} :=: e^{i\Delta^{(2)}_\mu(p,p')\widehat{z}^\mu + \Delta^{(2)}_\alpha(\eta,\eta')\widehat{\theta}^\alpha} : \quad (3.8)$$

where

$$\begin{array}{rcl} \Delta_0(p,p') &=& p_0 + p'_0 \\ \Delta_i(p,p') &=& p_i + e^{-\frac{p_0}{\kappa}} p'_i \\ \Delta_\alpha(\eta,\eta') &=& \eta_\alpha + e^{-\frac{p_0}{2\kappa}} \eta'_\alpha \end{array} \quad (3.9)$$

The κ–deformed Fourier supertransform can be defined as follows:

$$\Phi(\widehat{z},\widehat{\theta}) := \frac{1}{(2\pi)^2} \int d^4p\, d^2\eta\, \widetilde{\Phi}_\kappa(p,\eta) : e^{i(p\widehat{z} + \eta\widehat{\theta})} : \quad (3.10)$$

If we define inverse Fourier supertransform

$$\widehat{\Phi}(p,\eta) = \frac{1}{(2\pi)^2} \int d^4\widehat{z}\, d^2\widehat{\theta}\, \Phi(\widehat{z},\widehat{\theta}) : e^{-i(p\widehat{z} + \eta\widehat{\theta})} : \quad (3.11)$$

under the assumption that $(\widehat{\theta}^2 = \frac{1}{2}\widehat{\theta}_\alpha \widehat{\theta}^\alpha)$

$$\int d^2\, \widetilde{\theta}\, \widetilde{\theta}^2 = 1 \quad (3.12a)$$

or equivalently $(\eta^2 \equiv \frac{1}{2}\eta_\alpha \eta^\alpha)$

$$\frac{1}{(2\pi)^4} \iint d^4\widehat{z}\, d^2\widehat{\theta} :\, e^{i(p\widehat{z} + \eta\widehat{\theta})} := \delta^4(p) \cdot \eta^2 \quad (3.12b)$$

one gets

$$\widehat{\Phi}_\kappa(p,\eta) = e^{-\frac{4p_0}{\kappa}}\,\widetilde{\Phi}\left(e^{\frac{p_0}{\kappa}}\vec{p}, p_0, e^{\frac{p_0}{2\kappa}}\eta_\alpha\right) \tag{3.13}$$

For κ–deformed chiral fields one can consider their local powers, and perform the κ–deformed superspace integrals. One gets

$$\begin{aligned}
&\iint d^4\widehat{z}\, d^2\widehat{\theta} : \Phi(\widehat{z},\widehat{\theta}) = \widehat{\Phi}(0,0) \\
&\iint d^4\widehat{z}\, d^2\widehat{\theta}\Phi^2(\widehat{z},\widehat{\theta}) = \int d^4p_1\, d^4p_2\, d^2\eta_1\, d^2\eta_2 \qquad \text{(3.14a)}\\
&\widetilde{\Phi}_\kappa(p_1,\eta_1)\,\widetilde{\Phi}_\kappa(p_2,\eta_2)\,\delta(p_{01}+p_{02})\delta^{(3)}\left(\vec{p}_1 + e^{\frac{p_{01}}{\kappa}}\,\vec{p}_2\right)(\eta_1 + e^{\frac{p_{01}}{2\kappa}}\eta_2)^2 \\
&\iint d^4\widehat{z}\, d^2\widehat{\theta}\Phi^3(\widehat{z},\widehat{\theta}) = \int \prod_{i=1}^{3} d^4p_i\, d^2\eta_i \,\cdot\, \widetilde{\Phi}_\kappa(p_i,\eta_i) \\
&\quad \cdot\,\delta(p_{01}+p_{02}+p_{03})\cdot\delta^{(3)}\left(\vec{p}_1 + e^{\frac{p_{01}}{\kappa}}\,\vec{p}_2 + \frac{p_0+p_{02}}{\kappa}\,\vec{p}_3\right) \\
&\quad \cdot\,\left(\eta_1 + e^{\frac{p_{02}}{2\kappa}}\eta_2 + e^{\frac{p_{01}+p_{02}}{\kappa}}\eta_3\right)^2 \qquad \text{(3.14b)}
\end{aligned}$$

The formulae (3.14a)–(3.14b) can be used for the description of κ–deformed vertices in Wess–Zumino model for chiral superfields.

4 Star Product for κ–deformed Superfield Theory

In this section we shall extend the star product for the functions on κ–deformed Minkowski space given in [15] to the case of functions on κ–deformed chiral superspace, described by the relations (2.12a)–(2.12b).

The CBH $\star$–product formula for unordered exponentials takes the form

$$e^{ip_\mu z^\mu + \overline{\eta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}} \cdot e^{ip'_\nu z^\nu + \overline{\eta}'_{\dot\beta}\overline{\theta}^{\dot\beta}} = e^{i\gamma_\mu(p,p')z^\mu + \overline{\sigma}_{\dot\alpha}(p,p',\overline{\eta},\overline{\eta}')\overline{\theta}^{\dot\alpha}} \tag{4.1}$$

where

$$\begin{aligned}
\gamma_0 &= p_0 + p'_0 && \text{(4.2a)}\\
\gamma_k &= \frac{p_k\, e^{\frac{p'_0}{\kappa}} f\left(\frac{p_0}{\kappa}\right) + p'_k f\left(\frac{p'_0}{\kappa}\right)}{f\left(\frac{P_0+p'_0}{\kappa}\right)} && \text{(4.2b)}\\
\overline{\sigma}_{\dot\alpha} &= \frac{\overline{\eta}_{\dot\alpha}\, e^{\frac{p'_0}{2\kappa}} f\left(\frac{p_0}{2\kappa}\right) + \overline{\eta}'_{\dot\alpha} f\left(\frac{p'_0}{2\kappa}\right)}{f\left(\frac{P_0+p'_0}{2\kappa}\right)} && \text{(4.2c)}
\end{aligned}$$

and $f(x) \equiv \frac{e^x-1}{x}$. The star product multiplication reproduces the formula (4.1).

$$e^{ip_\mu z^\mu + \overline{\eta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}} \star e^{ip'_\nu z^\nu + \overline{\eta}'_{\dot\beta}\overline{\theta}^{\dot\beta}} = e^{i\gamma_\mu(p,p')z^\mu + \overline{\sigma}_{\dot\alpha}(p,p',\overline{\eta},\overline{\eta}')\overline{\theta}^{\dot\alpha}} \tag{4.3}$$

For arbitrary superfields $\phi(z,\theta)$ and $\chi(z,\theta)$ one gets

$$\phi(z,\theta) \star \chi(z,\theta) =$$

$$= \phi\left(\frac{1}{i}\frac{\partial}{\partial p_\mu}, \frac{\partial}{\partial \overline{\eta}_{\dot\alpha}}\right) \chi\left(\frac{1}{i}\frac{\partial}{\partial p'_\mu}, \frac{\partial}{\partial \overline{\eta}'_{\dot\alpha}}\right) e^{i\gamma_\mu(p,p')z^\mu + \overline{\sigma}_{\dot\alpha}(p,p',\overline{\eta},\overline{\eta}')\overline{\theta}^{\dot\alpha}} \Bigg|_{\substack{p=0\\ p'=0\\ \overline{\eta}=0\\ \overline{\eta}'=0}} \tag{4.4}$$

or equivalently

$$\begin{aligned}\phi(z,\theta) \star \chi(z,\theta) \quad = \quad & e^{iz^\mu\left(\gamma_\mu\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y'}\right) - \frac{\partial}{\partial y^\mu} - \frac{\partial}{\partial y'^\mu}\right) - \overline{\theta}^{\dot\alpha}\left(\overline{\sigma}^{\dot\alpha}\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial \omega}, \frac{\partial}{\partial \omega'}\right) - \frac{\partial}{\partial \omega_{\dot\alpha}} - \frac{\partial}{\partial \omega'_{\dot\alpha}}\right)} \\ & \cdot\phi(y,\omega)\chi(y',\omega')\Bigg|_{\substack{y=y'=z\\ \omega=\omega'=\overline{\theta}}}\end{aligned} \tag{4.5}$$

In particular we get

$$\begin{aligned}
z^i \star z^j &= z^i z^j \\
z^0 \star z^i &= z^0 z^i + \frac{i}{2\kappa} z^i \\
z^i \star z^0 &= z^0 z^i - \frac{i}{2\kappa} z^i \\
z^i \star \overline{\theta}^{\dot\alpha} &= z^i \overline{\theta}^{\dot\alpha} \\
\overline{\theta}^{\dot\alpha} \star z^i &= z^i \overline{\theta}^{\dot\alpha} \\
z^0 \star \overline{\theta}^{\dot\alpha} &= z^0 \overline{\theta}^{\dot\alpha} + \frac{i}{4\kappa}\overline{\theta}^{\dot\alpha} \\
\overline{\theta}^{\dot\alpha} \star z^0 &= z^0 \overline{\theta}^{\dot\alpha} - \frac{i}{4\kappa}\overline{\theta}^{\dot\alpha} \\
\overline{\theta}^{\dot\alpha} \star \overline{\theta}^{\dot\beta} &= \overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta}
\end{aligned} \tag{4.6}$$

Star product $\circledast$ corresponding to the multiplication of ordered exponentials (3.6) takes the form:

$$\begin{aligned}& e^{ip_\mu z^\mu + \overline{\eta}_{\dot\alpha}\overline{\theta}^{\dot\alpha}} \circledast e^{ip'_\mu z^\mu + \overline{\eta}'_{\dot\alpha}\overline{\theta}^{\dot\alpha}} \\ & = e^{i(p_0+p'_0 z^0 + i(e^{\frac{p'_0}{\kappa}} p_\kappa + p'_\kappa) z^\kappa + (e^{\frac{p'_0}{2\kappa}} \overline{\eta}_{\dot\alpha} + \overline{\eta}'_{\dot\alpha})\overline{\theta}^{\dot\alpha}}\end{aligned} \tag{4.7}$$

The superalgebra (2.12a)–(2.12b) of κ–deformed superspace is obtained from the following relations:

$$
\begin{array}{rcl}
z^k \circledast \overline{\theta}^{\dot{\alpha}} & = & \overline{\theta}^{\dot{\alpha}} \circledast z^k = z^k \overline{\theta}^{\dot{\alpha}} \\
\overline{\theta}^{\dot{\alpha}} \circledast \overline{\theta}^{\dot{\beta}} & = & \overline{\theta}^{\dot{\alpha}} \overline{\theta}^{\dot{\beta}} \\
z^k \circledast z^i & = & z^k z^i \\
z^0 \circledast z^i & = & z^0 z^i \\
z^i \circledast z^0 & = & z^0 z^i - \dfrac{i}{\kappa} z^i \\
z^0 \circledast \overline{\theta}^{\dot{\alpha}} & = & z^0 \overline{\theta}^{\dot{\alpha}} \\
\overline{\theta}^{\dot{\alpha}} \circledast z^0 & = & z^0 \overline{\theta}^{\dot{\alpha}} - \dfrac{i}{2\kappa} \overline{\theta}^{\dot{\alpha}}
\end{array}
\tag{4.8}
$$

Similarly like in nonsupersymmetric case the star–product (4.7) is more physical because reproduces the composition law of even and odd momenta consistent with coalgebra structure.

5 Final Remarks

In this lecture we outlined present status of quantum deformations of space–time supersymmetries[3], and for the case of κ–deformation of $D = 4$ supersymmetries proposed the corresponding deformation of chiral superfield theory. It appears that only the κ–deformed chiral superspace generators describe a closed subalgebra of κ–deformed $D = 4$ Poincaré group. At present it can be obtained the κ–deformation of superfield theory on real superspace can be obtained. The deformation of chiral superfield theory can be described by the following diagram:

The star product $\circledast$ given by formula (4.7) (see 4 on Fig. 1) is selected by the choice of superFourier transform (3.10), with ordered Fourier exponential described by (3.6). Equivalently, the $\circledast$–product multiplication can be obtained by the following three consecutive steps:

i) Deformation of local superfield theory (see 1 on Fig. 1)
ii) κ–deformed superfield transform (3.10) (see 2 on Fig. 1)
iii) inverse classical Fourier transform (see 3 on Fig. 1)

$$
\Phi(z,\theta) = \frac{1}{(2\pi)^2} \int d^4p \, d^2\theta \, e^{-i(p_\mu z^\mu + \eta_\alpha \theta^\alpha)} \widetilde{\Phi}(p,\eta) \tag{5.1}
$$

obtained in the limit $\kappa \to \infty$ from the inverse Fourier transform (3.11).

Finally it should be observed that for the deformation (1.1) with constant $\widehat{\theta}_{\mu\nu}$ there were calculated some explicite corrections to physical processes, in particular for $D = 4$ QED [29]–[31]. We would like to stress that these calculations should be repeated for Lie algebraic deformations of space–time and superspace, in particular in the κ–deformed framework. The preliminary results in this direction has been obtained in [32, 33].

[3]We did not consider here however, the quantum deformations of infinite – parameter superconformal symmetries in $1 + 1$ dimensions, described by superVirasoro algebras as well as affine $OSp(N; 2)$–superalgebras

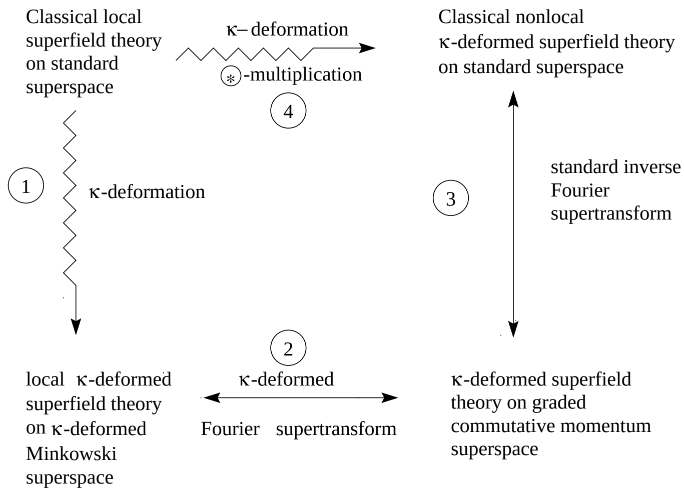

Figure 1: κ–deformation of local superfield theory

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